Market Signaling with Frictions[*]

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Abstract

In this paper, we study the signaling of qualities in a marriage model with frictions. This can lead to multiple equilibrium patterns. One interesting equilibrium outcome is that heterogeneous individuals choose identical actions to signal their quality, where this uniformity does *not* in any way depend on 'harsh' beliefs preventing deviations to actions off the equilibrium path. This equilibrium survives strong refinements which normally do not allow any pooling to survive. In the resulting equilibrium outcome, we observe perfect segregation of society in endogenously determined groups (classes), where individuals of each class conform to the same action and match exclusively with each other. Importantly, this behavior can also occur when the fundamentals allow only matching without classes in equilibrium if qualities were observable (following Smith 2002). In the setup with signaling, perfect segregation seems to occur more generally.

With the same fundamentals and preferences, a second equilibrium outcome can arise in which signals are chosen in such a way that *all* relevant information is transmitted, despite the frictions. Both equilibria survive the D1 criterium, while dissimilar equilibria are refined away. We argue that the mechanism behind the perfect segregation equilibrium can provide an explanation for conformity observed in social interactions.

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1 Introduction

Frictions are often thought to play an important role in (socio-)economic reality. In particular, there has been much interest in the implications of search frictions in two-sided matching models. In these models, two agents of two different kinds meet, and a match between the two can only be realized when both accept the agent on the other side. If one party does not accept, the match is dissolved, and the next opportunity to match is the next meeting. For obvious reasons, these are often referred to as marriage models. Compared to matching models in frictionless environments, acceptance decisions in matches change with frictions, as individuals are now willing to accept a partner (slightly) less good than the best achievable match: an individual will now accept a range of types as partner. The effects of the frictions on the overall matching patterns (who matches whom, in equilibrium) have been studied extensively by, among others, Burdett and Coles (1997), Eeckhout (1999) and Smith (2002).

The most common setting in which this is studied, is one where every person agrees on the ranking of the types of the individuals on the other side. Search frictions crucially imply that an individual does not know all relevant information in the marriage market. One might know or infer the distribution of these types in the population, but crucially, one does not know how to contact the best achievable match. Instead, a candidate match is randomly drawn from the type distribution, sequentially, and at a cost. This cost can be either a direct cost, or a waiting cost. Here, it is the standard assumption that one can, upon meeting, perfectly observe the type of the other person. The analysis in the literature to date stops here: further informational issues are usually assumed away[1], In this paper we attempt to go beyond this, and consider the case where qualities are unobservable, but can be signaled using a continuous signaling technology. Our interest is in how this affects the equilibrium matching patterns, and which patterns of signals accompany the matching equilibria.

This focus is also interesting because the patterns of matching have been the subject of a debate. The initial literature mainly emphasized matching in classes (Burdett and Coles 1997, McNamara and Collins 1990, Bloch and Ryder 2000), where marriage only occurred between persons from the same class (perfect segregation). This occurs surprisingly, *even though qualities within a class can differ substantially*, while on the other hand, the lowest quality of a class is only slightly better than the highest qualities of the following (lower) class. This corresponds to the right panel of figure 2 (where x denotes the quality index of the men, and y of the women). Important for this 'perfect segregation' result is the independence of utility *up to a scale* of one's own type (logmodular utility).

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[1]An exception being Chade (2005), who delves into the case of imperfectly observable qualities (noise in the observation of the quality).

Smith (2002) has called this kind of block matching "knife-edge", and "pathological", since in the setup of the aforementioned papers, it depends on the property that everybody values identically the relative gains (risk) of waiting for a new draw, given the same acceptances. If the relative risk is made type-dependent, matching with perfectly observable qualities will no longer be in 'classes'. For a large(r) class of utility functions (log-supermodular utility functions[2]), matching will be strictly positive assortative (PAM; see the left panel of figure 2). Technically, strict PAM means that for any two matches between types in the interior of the type-intervals the two highest types and the two lowest types of the two matches, can be agreeably rematched; strictness means that the two new matches are in the *interior* of the matching set.)
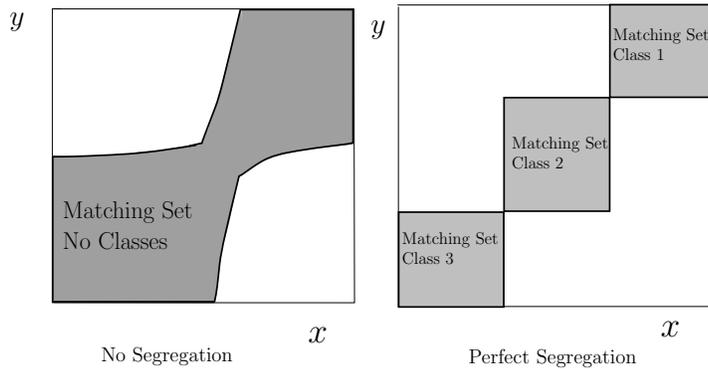

Figure 1: (A Form of) Strict Positive Assortative Matching and Perfect Segregation

However, in the setup of Smith, matching is in some sense passive. In the model, an individual is born with a perfectly observable quality. In the marriage market (with non-transferable utility) there is no active competition for a match: one is presented with a potential match, and one is accepted or rejected based on the quality that one is endowed with. The only question is how to *accept* optimally given these decisions of individuals on the other side[3]. Our interest, with signaling, is to give an individual a more active choice in terms of competition. In this paper, a person will be judged on his signal, not his type, and he can actually consider competing harder, by using a more expensive signal. Now, the frictions and the two-sidedness of the market can lead to multiple outcomes in terms of this 'active competition', in different equilibria. We will see that the active investment decision can again lead to perfect segregation, even in models where this does not happen in the case of perfectly observable qualities. However, in a different equilibrium, endogenously determined competition is harder than in the perfect segregation case, and individuals are forced to separate themselves

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[2]See assumption 2.

[3]That is, there is competition for matches, but —crucially— there is nothing one can do by himself to improve one's position.

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from all other individuals, and hence have different matching sets.

It deserves emphasis that the class formation does not depend in any way on choosing beliefs off the equilibrium path that induce 'harsh' treatment. Indeed, a nice feature of this model is that the perfect segregation equilibrium survives a strict refinement, the D1 refinement, that usually does not allow any class formation, or pooling. Here, however, class-formation happens endogenously, as indifferences in the *preferences* are generated endogenously, and thus not because of 'harsh' treatments for deviations.

Apart from the ramifications of the marriage model with frictions, we believe that the latter has some intuitive bearing on patterns of social interactions, which we will spell out in relative detail in section 5.

All in all, this paper attempts to make the point that frictions, combined with investment decisions, can again lead to segregation in classes. This segregation is thus no longer knife-edge. We see this here in a setting of signaling, where we show that the segregation result can arise endogenously. On the other hand, we also show that it not every equilibrium (surviving the refinements) is an equilibrium with perfect segregation. Indeed a second kind of equilibrium exists, where all relevant information is revealed, and the matching set looks like the case of perfectly observable qualities. It is a feature of the model that both equilibria exist side by side, and survive the D1 criterion side by side.

2   The Model

Our model is grounded in the literature on marriage models with frictions, most notable Burdett and Coles (1997), Eeckhout (1999), and Smith (2002).

One of the main motivations of this paper is to show the reappearance of a perfect segregation equilibrium when signaling is allowed, even in settings where the class structure does not arise with perfect observability with. Thus, we start with the particulars of Smith (2002), which have a strict positive assortative matching as outcome with perfectly observable qualities; the graph on the left of figure 2 corresponds to this case. Besides this interest, there are two more reasons for using Smith's assumptions. Positive assortativeness (in some average sense) seems an appropriate description of the marriage market and numerous other cases of human interaction. The line of reasoning used with PAM in this paper does also seem to be largely applicable to different utility specifications (when the appropriate changes are made). For example, in a setup with negative assortative matching (NAM), we conjecture that our results still could hold, after the appropriate relabeling[4].

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[4]Shimer and Smith use this trick as well for NAM. Our reasonings are applicable to many other specifications of utility as well, although some complications can arise. For example, some types find a gain in signaling, not because they will be treated better by higher types on the other side, but because

2.1 Primitives of the Model

People are divided into two categories, men and women. A person on one side gets utility from matching with ('marrying') a person on the other side, and vice versa. Men and women are treated symmetrically in the model, so in the coming analysis we can interchange the words 'men' and 'women'. Each man and woman is endowed with a quality. Concretely, there is a continuum of men, each indexed by his quality x, $x \in X = [\underline{x}, \bar{x}]$. Likewise, there is a continuum of women, each with quality $y \in Y = [\underline{y}, \bar{y}]$. Qualities are distributed continuously according to distributions $F_m(x), F_w(y)$, with full support over the closed and bounded interval $[\underline{x}, \bar{x}], [\underline{y}, \bar{y}]$. The women (men) rank the qualities of the men (women) in the same way (vertical heterogeneity), but they differ in the cardinal valuation of each man's (woman's) quality, according to their own type. In other words, women have a utility function $v(y, x)$ (the first variable denotes one's own type, the second variable denotes the type of the partner), assumed to be differentiable, with $v_2(y, x) > 0$. Likewise for men, utility is $u(x, y)$, with $u_2(x, y) > 0$.

Importantly, in order to consume a match, agreement from both sides is needed. Furthermore, we model this as a game of non-transferable utility: one side cannot overcome the reluctance of a preferred partner on the other side by making side-payments. This captures the reality that usually we cannot pay, or commit to pay, others to like us enough to enter long-term relationships.

As explained, the process of finding a match is not frictionless. In particular, we cannot make instantaneous offers to every possible candidate deemed good enough. In the model, agents only meet one candidate at a time, with stochastic arrival rate α, upon which they have to decide whether to make an offer or not. If both parties make an offer, they form a match; if not, the pair splits up, and each person has to wait till the next meeting occurs. Future gains are discounted, so there is clearly a cost of time, and a loss caused by the friction.

Although the ordering of qualities is agreed upon by everyone, the quality itself is not observed in a meeting. Instead, before entering the marriage market, each person can invest in a costly signal that could help convey information about their quality to the other person. The signal a can be chosen from a *continuous* interval of potential signals $[0, \bar{a}]$. We assume throughout the paper that the upper bound never becomes binding; this will have no effect on the types of results we derive. Initially, no one is endowed with a positive amount of the signal. This constitutes a major difference with the investment in self-improvement in Burdett and Coles (2001). Overall utility is given by the expected discounted lifetime utility from matching minus the disutility from investment in the signal (i.e. the two parts are additively separable).

Once matched (married), people leave the market, and are replaced by identical

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they would avoid matching each other.

unmatched individuals. This 'cloning' assumption is made for convenience, but also because it does not affect any of our conclusions in a qualitative way. In particular, in the steady state of a model with given inflows of types, individuals behave as though there were the cloning assumption, *given the steady state distribution.* Once matched, the marriage lasts forever; there is no on-the-job search or endogenous separation[5]. Also, polygamy is ruled out.

In short, life is three-staged.

- In the first stage, the agent is born, observes her quality, and decides to invest in the signal a. The action choices are denoted by $\alpha_m : X \to [0, \bar{a}], \alpha_w : Y \to [0, \bar{a}]$.
- In the second stage the agent enters the marriage market, and searches till she finds an acceptable partner to whom she is acceptable too. The acceptance functions are $\pi_m(x, a_m, a_w) \to \{0, 1\}$, $\pi_w(y, a_w, a_m) \to \{0, 1\}$, where 0 is rejection, 1 is acceptance. Note that a man's own signal is important for acceptance decisions of the women he meets, because it specifies his outside opportunities.
- Subsequently, when both parties, a man x with signal a_m, and a woman y with signal a_w accept each other (i.e. $\pi_m(x, a_m, a_w) \cdot \pi_w(y, a_m, a_w) = 1$) they marry and leave the market forever.

We will focus on pure strategies. A strategy in this game is, thus, a list of functions

$$\{a_m(x), \pi_m(x, a_m, a_w)\}, \ \{a_w(y), \pi_w(y, a_w, a_m)\} \forall \ x, y, a_m, a_w$$

In the subsequent analysis, we concentrate on the steady state, in which strategies of all individuals are stationary. Hence, the distribution of observed signals does not change from period to period.

2.2 Assumptions of the Model

We will focus on the case that yields positive assortative matching without classes when qualities would have been perfectly observable, and investigate what patterns of matching-on-signaling will result when qualities are made unobservable, but can be signaled.

First, let the utility functions satisfy the following regularity conditions

Assumption 1 (Regularity) *Utility $u(x, y)$, $v(y, x)$ is continuous and strictly positive on a compact set $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$, $\mathcal{C}^2$, and $u_2(x, y) > 0, v_2(x, y) > 0$ on the interior of this set. The value of being unmatched is denoted by $u(x, 0) = v(y, 0) = 0$.*

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[5]Arguably, this is a strong assumption in this setting, as true qualities cannot be observed till marriage, at which point it is too late. However, we conjecture that our reasoning about segregation and uniformity in signal still holds when divorce is costly (although this depends on specific assumptions on costs), or when there is a match-specific component to the realized utility of the match.

Moreover, we assume that utility is strictly log-supermodular. This assumption yields strict PAM in the case of perfectly observable qualities.

Assumption 2 (strict-LSPM) *Utility is strictly log-supermodular (on $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$), i.e. for $x_2 > x_1, y_2 > y_1$, $u(x_2, y_2)/u(x_1, y_2) > u(x_2, y_1)/u(x_1, y_1)$*

In this case, a strictly higher type will accept a strict subset of individuals on the other side which are accepted by the lower type (as will be proven later)[6]. Note that the log-supermodularity is basically relevant for weighing the relative benefits of accepting a current match with the gains of future matches; intuitively, if higher types have *relatively* more benefit from matching with higher types, they will hold out longer. What these values are in absolute terms does not matter in assumption 2. However, because there is an investment to be made in the first stage, in absolute terms (in common currency), we need to specify how utility behaves from type to type. First, let us define the costs:

Assumption 3 *Costs $C(x, a), C(y, a)$ are type- and side-independent, and given by $C(a) = c \cdot a$*

Since log(super)modularity is invariant to scalar multiplication, we need to assume additionally that the worth of matching with the lowest type is equal for all types or increasing in type in the *common currency* value, for own types. Thus, we have the following assumption[7]:

Assumption 4 $u_1(x, \underline{y}) \geq 0, v_1(y, \underline{x}) \geq 0$

Note that assumptions 2-4 define a single-crossing property, as in figure 2, where the trade-offs for types x_1 and x_2 are sketched between matching with a given type y for certain and paying for a signal a.

Crucially, with the utility from assumption 2, a higher type has more absolute benefit of matching with a given range of high types, and the absolute investment costs in a given signal are identical across types. Hence, it is relatively cheaper for high types to invest in (higher) signals, which results in the single-crossing property of figure 2. The reason for choosing single-crossing driven by utility, instead of driven by costs, which at first sight probably seems a more natural choice, is that we want to stay as close as possible to the setup of Smith (2002), and show that with signaling the two kinds of

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[6]Unless the lower bound of qualities of the other side is hit, in which case everyone is accepted. In that case, the matching set of individuals is still strictly increasing (in terms of set-inclusion) in one's own quality, for those particular qualities.

[7]We can replace the equality signs here with weak inequality signs. Together with strict log-supermodularity, this assumption boils down to assuming strict supermodularity:

Assumption 4b *Utility is strict supermodular, i.e. for $x_2 > x_1, y_2 > y_1$, $u(x_2, y_2) + u(x_1, y_1) > u(x_2, y_1) + u(x_1, y_2)$*
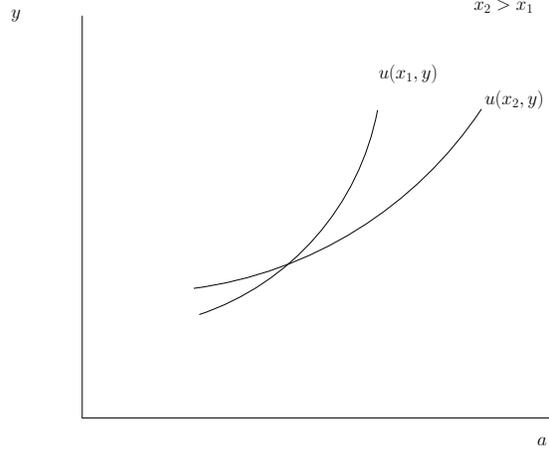
Figure 2: Single Crossing in this model

equilibria can and do occur in this setting. Here, because of the complementarities of qualities in the utility function, type-dependent costs are no longer necessary for single crossing[8].

2.3 The Game

As we mentioned, a person cannot observe the type of any potential partners, but only their previously made signaling choice.[9] Upon observing a signal from, say, a man, a woman decides to accept or reject him; to decide this, she takes into account her own utility (which is type-dependent), her inference about the possible types giving the signal she just observed, and her outside option.

   A woman thus infers a distribution of men who, she thinks, chose the observed signal a_m: $\tilde{G}_w(x|a_m)$[10] and bases her decision on the expectation over this distribution. In addition, for her outside option she will need to think about how well she would be accepted by the men based on the signal she chose, if she rejects. These acceptance decisions by the men depend on how the *other women* are choosing signals. Finally, it also matters for her outside option how the *men* are divided among signals. This

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[8]Note that assigning identical utility functions and type-dependent costs, as may seem more natural, would also lead to single crossing. However, have identical utility functions for every type implies that these are log-modular. As discussed, with this type of utility functions, class segregation arises even with observable types - a setup less interesting in this context.

[9]Here we assume an investment choice which is done before entering the marriage market. An alternative setup would be that *before each meeting*, a person has to invest in a signal. In equilibrium results will not differ qualitatively from the current setting. What is essential of the timing is that investment happens before meeting.

[10]We have chosen to work with conditional distribution here, because then we can interpret them as the beliefs about actions that do not occur in equilibrium (i.e. have a zero probability in the joint distribution).

affects how well the woman in question can distinguish types, and it also affects the acceptance decision of men.

At first sight these numerous interactions seem to make this problem very complicated, even more so since we have to keep track of distributions. Concretely this means that a person makes decisions based on the conjectured distribution function(s) $(\tilde{G}_m(y|a),\ \tilde{G}_w(x|a))$, *defined for every a*, the conjectured cumulative density function over action choices $\tilde{H}_w(a),\ \tilde{H}_m(a)$[11] for those actions played in the equilibrium with some probability, and the conjectured acceptance decisions of the other side $\tilde{\pi}_w(y, a_w, a_m)$.

Fortunately, the powerful 'inductive' solution procedure from the bilateral search literature, such as in Burdett and Coles, is also applicable here to simplify this seemingly complex problem. Moreover, we have chosen to keep the signaling setup itself as simple as possible (by having, for example, a very ordinary strict single crossing property).

Let us define the value for a man of type x of being unmatched, and having chosen signal a_m in the first round. Given $\tilde{G}_m,\ \tilde{H}_w,\ \tilde{\pi}_w(y, a_w, a_m)$,

$$rV(x, a_m) = \max_{\pi_m}\left\{ \alpha \int_A \int_Y \pi_m(x, a_m, a_w)\tilde{\pi}_w(y, a_w, a_m) \cdot \right.$$
$$\left. \left[u(x, y) - V(x, a_m)\right] d\tilde{G}_m(y|a_w) d\tilde{H}_w(a_w) \right\} \qquad (1)$$

Likewise, for a woman of type y, having chosen a_w in the first round,

$$rW(y, a_w) = \max_{\pi_w}\left\{ \alpha \int_A \int_X \pi_w(y, a_w, a_m)\tilde{\pi}_m(x, a_m, a_w) \cdot \right.$$
$$\left. \left[v(y, x) - W(y, a_w)\right] d\tilde{G}_w(x|a_m) d\tilde{H}_m(a_m) \right\} \qquad (2)$$

Of course, in equilibrium the conjectures about decision rules and distributions must be fully rational, and thus coincide with the equilibrium decisions and distributions.

As is standard in the literature (see Eeckhout 1999 for example), we assume that a person always accepts offers if it is weakly to her advantage to accept; this rules out equilibria where the best man does not accept the best woman because she does not accept him, and vice versa.

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[11]Technically, these functions are a mix of cumulative density and mass functions, since some (zero mass) actions can occur with strictly positive probability.

Lemma 1 *The optimal decision rules satisfy*

$$\pi_m(x, a_m, a_w) = \begin{cases} 1 & \text{if } \int u(x, y) d\tilde{G}_m(y|a_w) \geq V(x, a_m), \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\pi_w(y, a_w, a_m) = \begin{cases} 1 & \text{if } \int v(y, x) d\tilde{G}_w(x|a_m) \geq W(y, a_w), \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Proof. Straightforward from the value function, expected utility maximization, and the assumption to always accept when indifferent ∎

From now on we will not repeat arguments and definitions for both women and men, when they apply to both sides of the market. It is assumed that the definitions, and requirements are mirrored on the other side.

Note that we also have to specify the reaction to off-equilibrium path actions, based on beliefs $\tilde{G}_m(y|a_w)$. [12]

In the first stage of the game, each man chooses his signaling action, to maximize his ex ante payoff:

$$\alpha_m(x) = arg \max V(x, a_m) - C(x, a_m). \tag{5}$$

$\alpha_w(y)$ is derived similarly for each woman. Note that $\alpha_w^{-1}(a), \alpha_m^{-1}(a)$ define a correspondence, not necessarily a function.

Now, we can define an equilibrium in the game.

Definition 1 *For given distributions F_w, F_m, a sequential 'matching-on-signals' equilibrium (MOSE) is a list of signaling rules $\alpha_m(x)$, $\alpha_w(y)$, and acceptance rules $\pi_m(x, a_m, a_w)$, $\pi_w(y, a_w, a_m)$; cumulative density functions $H_m(a_m)$, $H_w(a_w)$, and conditional distributions $G_m(y|a_w)$, $G_w(x|a_m)$; and the associated value functions $V(x, a_m)$ and $W(y, a_w)$, such that $\forall x, y$*

1. *(Second-stage optimization) Given π_w, G_m, H_w, $\pi_m(x, a_m, a_w)$ and $V(x, a_m)$ are given by (3) and (1); similarly $W(y, a_w)$, $\pi_w(y, a_w, a_m)$ are given for the women.*
2. *(First stage optimization) $\alpha_w(y)$ is given by (5), given $W(y, a_w)$. Similarly for the men.*
3. *(Consistency of H) $H_w(a_w)$ is given by*

$$H_w(a_w) = \int_0^{a_w} \left(\int_{\alpha^{-1}(\chi)} dF_w(y) \right) d\chi \tag{6}$$

The definition for $H_m(a)$ is symmetric.

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[12]Suppose a man observes an action x_w which is not played in equilibrium. He must have beliefs about the possible types that would do this (zero-mass) deviation, *and* strictly speaking he must also have a belief about the acceptance decision of these types! However, as the men is not worse off if he accepts, while the deviating woman rejects, this is not of concern to us here.

4. (Equilibrium-path consistency of G) *On the equilibrium path, beliefs are consistent (derived from Bayes' Rule)*

(a) *if supp $\alpha^{-1}(a_m)$ is not of measure zero:*

$$G_w(x|a_m) = \frac{f_m(x)}{\int_{\alpha^{-1}(a_m)} dF_j(x)} \tag{7}$$

Analogous for $G_m(y|a_w)$.

(b) *if supp $\alpha^{-1}(a_m)$ is of measure zero, it is in this case sufficient to assume:*

$$supp\ G_w(x|a_m) = \ supp\ \alpha^{-1}(a_m) \tag{8}$$

The third and fourth condition together tell us that in equilibrium, agents conjecture the right joint distribution of types and actions. On other words, they hold the right beliefs about how people of different types are distributed along the signaling spectrum, in equilibrium[13]. In this we assume that individuals of identical types choose identical actions.

3   Equilibria with Observable Qualities

We now take a step back, and look —in more detail— at the equilibria in the marriage model with frictions but perfectly observable qualities. The aim of this exercise is two-fold. We want to contrast the observable case with the case of signaling, but in the process we also derive some of the concepts and properties needed for the case of unobservable qualities and signaling. Note that the case of perfect observability completely follows the setup above, where we make the appropriate substitutions for 'beliefs' and drop the argument for the signaling action.

Lemma 2 (Reservation Types) *If qualities were perfectly observable, the optimal policy would be a 'reservation type' policy*[14]

Proof. This follows trivially from lemma 1 in case each type plays a distinct action: $u(x, y) \geq V(x)$ ∎

Corollary 1 *Higher types are accepted by more individuals on the other side than lower types*

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[13]Note that there always exists an equilibrium in which no one invests in the signal (i.e. $\alpha_m(x) = 0 \forall x, \alpha_w(y) = 0\ \forall\ y$ and $\pi_m(x, a_m, a_w) = 1, \pi_w(y, a_w, a_m) = 1\ \forall\ x, y, a_m, a_w$), and everyone matches with everyone (assuming that it is better to match with the lowest type than to remain single). Here the key issue is that it does not pay to distinguish oneself from the masses, as everyone is accepted anyway. This reasoning will play a role against the 'drive to set oneself apart' throughout this paper.

[14]Given that a match is accepted always when it is weakly optimal to do so.

Definition 2 *Denote by $\gamma(x)$, $\delta(y)$ the acceptance functions, or reservation functions, conditional on the fact that everyone on the other side accepts everyone.* *Suppose* $\bar{\pi}_m : (x, y) \rightarrow \{0, 1\}$, *and*

$$r\bar{V}(x) = \max_{\bar{\pi}_m(x,y)} \alpha \int_{\underline{y}}^{\bar{y}} \bar{\pi}_m(x, y) \left[u(x, y) - \bar{V}(x) \right] dF_w(y).$$

Then, by lemma 2, $\bar{\pi}_m$ is monotone. Let $\gamma(x) \equiv \min\{y | \bar{\pi}_m(x, y) = 1\}$.[15] *Likewise for the women.*

    Let $\Gamma(x)$, $\Delta(x)$ be the equilibrium acceptance functions (given the equilibrium acceptance decisions of the other side). These are simply defined from equilibrium $\pi_m(x, y)$, $\Gamma(x) \equiv \min\{y | \pi_m(x, y) = 1\}$, and likewise for the women.

We assume that $\gamma(\underline{x}) > \underline{y}$, $\delta(\underline{y}) > \underline{x}$; i.e. all individuals are sufficiently patient, so that they would not accept everybody when they are accepted by everybody themselves[16]. Now we can show the following proposition.

Proposition 1 *Given assumptions 1 and 2, $\gamma(x), \delta(y)$ are continuous, strictly increasing functions*

    To prove this, first fix a type x. For this x, the problem becomes a relatively simple decision-theoretic one. Let $\beta \equiv \frac{1}{1+r}$; clearly $\beta < 1$ for any $r > 0$. Rewriting discounting in terms of β is more convenient to derive the properties below. Assume $\alpha\beta < 1$ throughout (with the appropriate rescaling of time units, we can do so without loss of generality). Define a mapping T as follows:

$$TV_x \equiv \alpha\beta \int_{\underline{y}}^{\bar{y}} \max\{u(x, y), V_x\} dF_w(y) + (1 - \alpha)\beta V(x) \tag{9}$$

We can now look for a fixed point of T. This approach is a convenient simplification; looking for value functions as fixed points in function space can be relatively complex, as under general assumptions the equilibrium value function might not be continuous.

Lemma 3 *T defines a contraction; moreover, given V^0, $\{T^n V^0\}$ defines a sequences which converges monotonically towards the fixed point V^*.*

Proof. Define $\bar{V}, \hat{V}$, with $\bar{V} \geq \hat{V}$. Let $y_1 = \min\{y | u(x, y) = \bar{V}\}$ and $y_2 = \min\{y | u(x, y) = \hat{V}\}$. Since we have that for $y_1 \geq y \geq y_2$ $\bar{V} \geq u(x, y) \geq \hat{V}$, we have that

$$\alpha\beta \int_{y_2}^{y_1} (\bar{V} - \hat{V}) dF_w(y) > \alpha\beta \int_{y_2}^{y_1} (u(x, y) - \hat{V}) dF_w(y).$$

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[15]By continuity of u the minimum exists.

[16]This only means that r cannot be too large. The purpose is just to focus on the interesting case, as the case where everybody matches nearly everybody in equilibrium, because impatience is huge, does not enlighten much.

which leads to the following inequality

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\begin{aligned}
\bar{V} - \hat{V} &> \alpha\beta(\bar{V} - \hat{V}) + (1-\alpha)\beta(\bar{V} - \hat{V}) \\
&> \alpha\beta \int_{y_2}^{y_1} (u(x,y) - \hat{V})dF_w(y) + \alpha\beta \int_{\underline{y}}^{y_2} (\bar{V} - \hat{V})dF_w(y) \\
&\quad + (1-\alpha)\beta\bar{V} - (1-\alpha)\beta\hat{V} \tag{10} \\
&= T\bar{V} - T\hat{V} > 0 \tag{11}
\end{aligned}
$$

which establishes the contraction property *and* the monotonicity of convergence. ∎
This means that $\gamma(x)$ is well-defined (uniquely, given the regularity conditions), for every x, and is a solution of a simple decision theoretic problem: $\gamma(x) = \{y \in \mathbb{R} | Ty = y\}$.

Proof of Proposition 1.   We want to prove that, if $x_2 > x_1$, then $y_2 \equiv \gamma(x_2) > \gamma(x_1) \equiv y_1$. Let $V_{x_1} = u(x_1, y_1)$; and call $\Omega \equiv \frac{\alpha\beta}{1-\beta+\alpha\beta}$ Then, bring the last RHS-term of (9) to the LHS, and solve for V_{x_1} to get

$$
\begin{aligned}
V_{x_1} &= \Omega \int_{\underline{y}}^{\bar{y}} \max\{u(x,y), V_{x_1}\}dF_w(y) \tag{12} \\
&= \Omega \left(\int_{y_1}^{\bar{y}} u(x_1, y)dF_w(y) + u(x_1, y_1)F_w(y_1) \right) \tag{13}
\end{aligned}
$$

Now, we need to show that for $V_{x_2} \equiv u(x_2, y_1)$, it holds that $TV_{x_2} > V_{x_2}$.
   Pre-multiply equation (13) by $\frac{1}{u(x_1,y_1)}$, to get

$$
1 = \Omega \left(\int_{y_1}^{\bar{y}} \frac{u(x_1,y)}{u(x_1,y_1)}dF_w(y) + F_w(y_1) \right). \tag{14}
$$

From log-supermodularity, the following inequality must hold

$$
\int_{y_1}^{\bar{y}} \frac{u(x_2,y)}{u(x_2,y_1)}dF_w(y) > \int_{y_1}^{\bar{y}} \frac{u(x_1,y)}{u(x_1,y_1)}dF_w(y). \tag{15}
$$

Then

$$
TV_{x_2} = \Omega \left(\int_{y_1}^{\bar{y}} u(x_2,y)dF_w(y) + u(x_2,y_1)F_w(y_1) \right). \tag{16}
$$

We are done when we can prove that $TV_{x_2} > u(x_2, y_1)$ ($= V_{x_2}$), or equivalently $TV_{x_2}/V_{x_2} > 1$, but this can be derived from comparing equations (14) and (16), with help of equation (15). Dividing (15) by $u(x_2, y_1)$, we get

$$
\begin{aligned}
\frac{TV_{x_2}}{V_{x_2}} &= \Omega \left(\int_{y_1}^{\bar{y}} \frac{u(x_2,y)}{u(x_2,y_1)}dF_w(y) + F_w(y_1) \right) \\
&> \Omega \left(\int_{y_1}^{\bar{y}} \frac{u(x_1,y)}{u(x_1,y_1)}dF_w(y) + F_w(y_1) \right) \\
&= 1
\end{aligned}
$$

13

Next, we show there can be no jumps (i.e., $\gamma(x)$ is continuous). Note first that since $\gamma(x)$ is increasing on a bounded interval, and $\gamma(\bar{x})$ is finite, it must be that both $\gamma(x)$ and $V(x)$ are of bounded variation, and differentiable almost everywhere. There are at most a countable number of discontinuities.

Let $\tilde{x}$ be a point of discontinuity in $\gamma(x)$, and let $\tilde{y} = \gamma(\tilde{x})$. Because, as we established, it is increasing, the function can only jump upwards. Since there are at most a countable number of discontinuities, we can take the limit from the right

$$\lim_{x \uparrow \tilde{x}} \gamma(x) \equiv \gamma(\tilde{x}_{lim}).$$

Define $V_{lim}(\tilde{x})$ to be $u(\tilde{x}, \gamma(\tilde{x}_{lim,}))$. Then we can find a $\delta > 0$ such that

$$V_{lim}(\tilde{x})) < V(\tilde{x}) - \delta.$$

This implies that for $\varepsilon > 0$ very small, $V(\tilde{x} - \varepsilon) < V(\tilde{x}) - \delta$. By continuity of $u(x, y)$, and therefore the continuity of the integral for a fixed acceptance interval, and given that indeed the *opportunities* for all types are the same, we have that for ε small enough, a type $\tilde{x} - \varepsilon$ copying the acceptance policy of $\tilde{x}$, will yield $V(\tilde{x}) - \eta > V(\tilde{x}) - \delta$, contradicting optimality. ■

Note that this proof differs from Smith (2002); he derives only almost everywhere differentiability. His setup is different, but our proof applies to his results as well.

Now, we can focus on the acceptance decisions $\Gamma(x), \Delta(y)$ in equilibrium. Following Eeckhout (1999) for the existence, we have:

Proposition 2 *Given assumptions 1 and 2, an equilibrium exists. In an equilibrium, the acceptance functions $\Gamma(x), \Delta(y)$ are continuous and strictly increasing.*

The proof is in the appendix. Note that no segregation will occur in this case. The matching graph can look similar to the graph on the left, in figure 2; for sure, it cannot look like the graph on the right in this figure. Furthermore, the acceptance function relates to the value functions, so we can derive the following:

Lemma 4 *With assumptions 1 and 2, value functions $V(x)$ and $W(y)$ are continuous, and differentiable almost everywhere*[17].

Proof. Given assumption 1 and 2, value functions are increasing and bounded, and hence are of bounded variation. Thus they are differentiable a.e. Continuity follows immediately from the continuity of both $\Gamma(x)$ and $\Delta(y)$ (In fact, the two continuities were both shown in the proof of proposition 2.) ■

In general, we have the following:

---

[17] Generically there are kinks at $x_1, x_2, \ldots$, where $x_1 = \Delta(\bar{y})$, and $y_1, y_2, \ldots$, where $y_1 = \Delta(\bar{x})$; and subsequent $x_2, \ldots; y_2, \ldots$, defined inductively by $x_i = \Gamma(y_{i-1}), y_i = \Delta(y_{i-1})$.

Proposition 3 *Given assumption 1, and let utility be (weakly) log-supermodular. An equilibrium exists. However, a perfectly segregated class can only occur if the utility functions of all individuals in this class are logmodular,* and if the utility functions from *every* individual with a *higher* quality is logmodular *(We disregard zero measure deviations)*

The proof is in the appendix.

4   Equilibria in the Signaling Game

Frictions lead men to accept a range of types of women, and vice versa. However, when these qualities have to be inferred from signals, the interaction between these and the the frictions becomes interesting. For one, the information value of all other signals, together with the range acceptance can lead to different choices when deciding on your own investment in the signal in the first stage, and on whom to accept in the second stage. Below, we derive the properties of equilibria, and describe the different kinds of equilibria in this game. First, we are interested in a very natural property that is general to all signaling equilibria in this setup (but requires a few technical steps), which will help us with characterization. Secondly, in signaling games there is usually a vast multitude of equilibria, of which only some are intuitive and interesting. Those equilibria are commonly selected by applying equilibrium refinements to the set of equilibria. We need to do this here, too: in some sense our setup is a 'signaling game squared'[18], and has the 'squared' multitude of equilibria that comes with it. We apply the natural version for this context of the D1 refinement to the set of equilibria, and find that two kinds of symmetric equilibria survive, the perfect segregation and the separation equilibrium. After laying out the refinement, in the last two parts of this section we will construct the two different equilibria.

4.1   Properties of the Equilibria

When, in an equilibrium, people pool on an action, there are two main complications, compared to the above model with perfectly observable qualities. For one, the distribution of expected utilities based on observable characteristics now endogenously has mass points in it. Secondly, the utility value of these mass points differs from type to type. What we are looking for is the same regularity in acceptance decisions as we had in the case with observable qualities: a reservation property. From lemma 1, we can translate the acceptance decision in terms of expected types, instead of expected utility, *if the set of types pooling on an action is connected*, which we show to be the case, and

---

[18]To be entirely correct it is even more than two simultaneous signaling games merged together, as the outside options are also determined in future expected signaling games.

subsequently derive a reservation property in actions played on the equilibrium path (the set of which we denote by A_m^{eq}, A_w^{eq} for men and women, respectively)

Lemma 5 *Suppose assumptions 1 through 4 apply. In an equilibrium with pooling actions, the sets of types pooling are connected. The optimal policy is a reservation policy* with respect to the actions played in equilibrium. *In other words, for actions $a_1, a_2 \in A_m^{eq}$, such that $a_2 > a_1$, it now holds that*

$$\pi_w(y, a_w, a_1) = 1 \Rightarrow \pi_w(y, a_w, a_2) = 1 \quad \forall y, \forall a_w \tag{17}$$

Proof. First, we show that, under the above-mentioned assumptions and qualifications, an action played by higher quality types is accepted by more agents on the other side. Suppose, two actions a_1, a_2 are played by men in equilibrium, with $\mathbb{E}[x|a_2] > \mathbb{E}[x|a_1]$, *where in case of pooling, a connected set of types is choosing the same action.* It must hold for any y that $\mathbb{E}[v(y, x)|a_2] > \mathbb{E}[v(y, x)|a_1]$, for which the connectedness is essential. From the reservation property in expected types, established in lemma 1, it must be that

$$\pi_w(y, a_w, a_1) = 1 \Rightarrow \pi_w(y, a_w, a_2) = 1$$

for all y, a_w. This implies that if we compare the benefits of playing a_2 versus a_1, an agent playing a_2 gets treated at least as well as playing a_1. Thus, it must hold that $V(x, a_2) \geq V(x, a_1)$, $\forall x$.

In an equilibrium with $a_2 \neq a_1$, and both actions are played, it must be that $V(x, a_2) > V(x_1)$.

Then, secondly, we need that only higher types find it profitable to play a_2, given that $V(x, a_2) > V(x, a_1)$, $\forall x$. The important step is to show that the relative gain of playing a_2 instead of a_1, $V(x, a_2) - V(x, a_1)$, is increasing in type x.

To prove this, suppose we have x_1, x_2, with $x_2 > x_1$. Note that if two agents take the same action, trivially their opportunities (i.e. which types are accepting *them*) must be the same. However, for the two actions a_2 and a_1 to be played in equilibrium, there must be a strict difference in the respective opportunities associated with these actions. Given the assumption $\mathbb{E}[x|a_2] > \mathbb{E}[x|a_1]$, those taking a_2 are accepted by all types who accept a_1 (by the first step), *plus a strictly positive mass of other types.* Crucially, among these other women there must be a non-zero measure subset playing actions in equilibrium which provide an average(!) expected utility to x_1 higher than the $V(x_1, a_1)$. To see this, suppose to the contrary that this is not the case. Given identical opportunities, higher types of men are pickier, from proposition 1. Hence, if x_1 does not want to accept these women, neither will any $x > x_1$. Then, these higher quality men will not play a_2, and $\mathbb{E}[x|a_2] \leq \mathbb{E}[x|a_1]$. Contradiction.

Denote by A_w, the set of actions by women who are accepting those playing a_2, *but do not accept those playing a_1*, and are *also accepted by x_1 playing a_2*. In other

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words: $A_w = \{a_w | \pi_w(y, a_w, a_1) = 0 \wedge \pi_w(y, a_w, a_2) = 1 \wedge \pi_m(x_1, a_w, a_2) = 1, \ \forall y\}$. (In the construction of set A, it is important that x_1 is accepting optimally at a_2.) Player x_2 can also accept this set A, and possibly do even better (by not accepting those who do not give sufficient utility). We want to show that only based on those with actions in A, x_2 already gets strictly more utility than x_1. In other words, x_2 strictly profits more from this increased acceptance.

To see this, consider

$$\int_{A_w} \int_{\underline{y}}^{\bar{y}} u(x_1, y) dG(y|a_w) < \int_{A_w} \int_{\underline{y}}^{\bar{y}} \frac{u(x_2, y)}{u(x_2, \underline{y})} u(x_1, \underline{y}) dG(y|a_w) \tag{18}$$

by log-supermodularity; now, by assumption 4, we know that $u(x_2, \underline{y}) \geq u(x_1, \underline{y})$, so from (18) $\int_{A_w} \int_{\underline{y}}^{\bar{y}} u(x_1, y) dG(y|a_w) < \int_{A_w} \int_{\underline{y}}^{\bar{y}} u(x_2, y) dG(y|a_w)$.
Thus, we have

$$
\begin{aligned}
V(x_1, a_2) - V(x_1, a_1) &= \alpha\beta \int_{A_w} \int_{\underline{y}}^{\bar{y}} u(x_1, y) dG(y|a_w) \\
&< \alpha\beta \int_{A_w} \int_{\underline{y}}^{\bar{y}} u(x_2, y) dG(y|a_w) \\
&< V(x_2, a_2) - V(x_2, a_1),
\end{aligned}
$$

which was what we needed to show.

Our final step is to show that the property we have just derived implies that indeed pooling occurs in connected sets, in equilibrium. Moreover, we establish that higher types indeed play weakly higher actions. Thus, for $x \in \alpha^{-1}(a_2)$, $x' \in \alpha^{-1}(a_1)$, with $a_2 > a_1$, it must hold that $x > x'$. Suppose not. By optimality, $V(x', a_1) - ca_1 > V(x', a_2) - ca_2$, and $V(x, a_1) - ca_1 < V(x, a_2) - ca_2$. This implies that

$$V(x, a_2) - V(x, a_1) > ca_2 - ca_1 > V(x', a_2) - V(x', a_1)$$

contradicting the result that $V(x_1, a_2) - V(x_1, a_1) < V(x_2, a_2) - V(x_2, a_1)$ if and only if $x_1 < x_2$. Hence, if $x \in \alpha^{-1}(a_2)$, $x' \in \alpha^{-1}(a_1)$, with $a_2 > a_1$, then $x > x'$. This establishes the connectedness *and* that $\mathbb{E}[x|a_2] > \mathbb{E}[x|a_1]$ if $a_2 > a_1$, and therefore that, if $a_2 > a_1$

$$\pi_w(y, a_w, a_1) = 1 \Rightarrow \pi_w(y, a_w, a_2) = 1 \ \ \forall y, \forall a_w \tag{19}$$

∎

Note that in the proof of this lemma we have derived the following property:

Corollary 2 *If for actions a_1, a_2, with $a_2 > a_1$ and type x it holds that $V(x, a_2) - V(x, a_1) \geq c(a_2 - a_1)$, then for all $x' < x$, $V(x', a_2) - V(x', a_1) < c(a_2 - a_1)$*

Lemma 5 gives us the following important results. First, in any equilibrium a pooling group consists of a connected set of types. Moreover, in checking deviations, we only

have to do so for the highest and lowest members of the pooling group. The highest quality individuals will play the highest actions in equilibrium. All these properties help us characterize equilibria later on. However, none of these results should come as a big surprise; they all follow from the single crossing property established by assumptions 2 through 4.

4.2   Refinements

As could be expected, the reservation property of lemma 5 mirrors the full-information reservation type policy in Burdett and Coles (1997), Eeckhout (1999) and Smith (2002). This property, however, holds in the signaling case only for actions that are played in equilibrium: in principle, more expensive signals *off* the equilibrium path, can neverthe-less be rejected by the same individual. The reason is that in a sequential equilibrium, off the equilibrium path we can hold beliefs that these actions, even when the cost is very high, are played by unacceptably low individuals. (Since these actions are not played in equilibrium, we are able to hold these beliefs in Nash, even sequential, equi-librium, even if benefits of the best treatment possible would not outweigh the costs of the high signals.) Because of this, pooling can be 'enforced', by believing that any off-path deviator is a low type, and will be treated as such. No one, as a result, will deviate to off-equilibrium actions, even if being part of a pooling group is not very ad-vantageous either. This is hardly intuitive, as it does not acknowledge that the highest types have most to gain from deviating, in this setting, and looks also a bit artificial. The challenge therefore is to have pooling in multiple groups in an equilibrium where deviations are thought to come from higher types, those who have most to gain from deviating (upwards). The *D1 criterion* (refinement) restricts the off-equilibrium beliefs to precisely incorporate this. Moreover, in signaling games with strict preferences, D1 equilibria never exhibit pooling in multiple groups. We will prove that the equilibria exhibited below survive D1 below, after defining the D1 criterion.

We define the D1 criterion (following Ramey 1996), in a general signaling game with multiple types of senders, and one receiver. Let $V(a, r, x)$ be the utility of the sender, depending on his action a, the response of the receiver r, and his type x. Let $\mathbb{G} = \{\hat{a}(x), \hat{r}(a), \hat{G}(x|a)\}$ be a sequential equilibrium, with receiver's beliefs $\hat{G}(x|a)$ about the distribution of types taking action a. Let the sender's equilibrium payoff be denoted by

$$\hat{V}(x) \equiv V(\hat{a}(x), \hat{r}(a), x)$$

Definition 3 *Fix an action $a \notin$ range $\hat{a}$, and suppose we have a set of types $\mathbb{X} = [\underline{x}^{\flat}, \bar{x}^{\flat}]$ for which holds: $\forall \ x \notin \mathbb{X}, \exists \ x'$, s.t. $V(a, r, x) \geq \hat{V}(x) \Rightarrow V(a, r, x') > \hat{V}(x')$. This equilibrium violates the **D1 criterion** if supp $\hat{G}(x|a) \nsubseteq \mathbb{X}$.*

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This tells us, for a given action, if there is a type which always has a strict preference to deviate to this action when a second type has only a weak preference, we should place zero probability on the second type, in our beliefs about the identity of the deviator.

To be complete here, our game is not a signaling game in the strictest sense, which could imply that the D1 criterion has to be redefined in our setting. We argue that this is not necessary. Our game is in some sense a '(more) general equilibrium' of many signaling games played (the equilibrium outcomes of the other signaling games in the economy determine one's outside options, and one's own acceptance decision provides additional insurance). However, we can collapse the results of other signaling games in the 'general equilibrium', and put the, now taken as given, outside options in the reduced-form utility functions of both sender and receiver. In effect, in the newly defined partial equilibrium signaling game, all they care about is the gain, when positive, over the outside option. What is more, we can redefine the receiver to be an maximizer of utility before receiver type has realized. Now his decision is to find which types will accept given a certain message[19]. Likewise, we can take the ex ante expected utility for the sender, before his match has realized. Now, we have collapsed our game into an ordinary signaling game, with a continuum of sender types and one receiver, albeit with potentially strange utility functions. In this reduced form signaling game we can apply the common D1 criterion, without the need to redefine it. Thus, in the analysis below, we are always applying D1 while taking the outside options of both sides as given. (The outside options are part of the reason that both pooling or separation survive the D1 equilibrium)

At this stage, we conjecture that the only equilibria that survive this refinement are symmetric in the sense that the structure, be it class or separation, on one side is mirrored on the other side[20]. Now, we can investigate which kinds of equilibria there are in the signaling game.

4.3   Perfect Segregation Equilibrium

The essential property of perfect segregation is that one side wants to accept the mirroring class on the other side, and only that class; and vice versa. This points to a very simple relation of perfect segregation to reservation types: the reservation type of one class should be the lowest member of the other class, and vice versa[21]. Thus, for the

---

[19]This redefinition is without loss of generality

[20]With the possible exception of a trivial equilibrium, where one class does not distinguish itself at all

[21] Or, more generally, the reservation type of the lowest member of the class on one side should be *higher* than the lowest member of his class on the other side, and vice versa; we here concentrate on the case that they are equal to get maximal separation in classes; Moen (1997) uses a selection argument that can also be used here.

highest classes, who are accepted by everyone, it should hold that

$$\gamma(x_1^c) = y_1^c, \delta(y_1^c) = x_1^c \tag{20}$$

Since all are accepted, they will play the same action. Higher type members of a class may have a higher reservation type (in perfect information) but can simply not distinguish in equilibrium among relatively low types and high types within the class, and have to accept all in the class.

If a class defined by this equation separates itself, it will be very stable, in the sense that it will not accept anyone else, even if a person reveals himself to be the best of the rest. Moreover, we can simply find the next class by deleting the first class from the distribution, thereby adjusting the chance of meeting non-first class individuals. If the same kinds of cutoffs are chosen for the next class, they will also not accept anyone else than the first two classes, even when —again— a person reveals himself to be best of the rest. We can see that this equilibrium will survive the D1 criterion precisely for this reason.

Having said all this, we can completely spell out the equilibrium with perfect segregation.

Proposition 4 *There exists at least one equilibrium with perfect segregation*

Proof. By construction. Assume A, the signaling 'spectrum', is large enough. We define the first class on each side, by its lowest member, x_i^c, where i stands for the number of the class, counting from the highest class.

• STEP 1: Let us determine the first class by $\gamma(x_1^c) = y_1^c, \delta(y_1^c) = x_1^c$.

Now the highest class on each side for sure only want to accept each other. This point exists, by (as a sufficient, not necessary condition) continuity; see figure 3^{22}

• STEP 2: Delete the first class from the distribution. Take into account that the chance of meeting someone not in the first class is lower than the original α meeting rate, and that the distribution of the remaining types has to be normalized to one again[23]; and repeat the procedure. Repeat the procedure until $x_i^c = \underline{x}$, or $y_i^c = \underline{y}$. Potentially, there can be a group of unmatched agents on *one* side. Call this the lowest class, I, otherwise, the lowest matching class is I.

• STEP 3: Choose the signaling actions that belong to the lowest class. Suppose a group of men is unmatched. Then, for $\underline{x} \leq x < x_{I-1}^c, \alpha_m(x) = 0 \equiv a_{mI}$. (For $\underline{y} \leq y < y_{I-2}^c, \alpha_w(y) = 0$; if both last classes match each other, replace y_{I-2}^c by y_{I-1}^c)

---

[22]Note that if r is very high, both $\delta(y)$ and $\gamma(x)$ might be on the lower bound $\underline{x}, \underline{y}$, for most x, y. Conversely, we can have a lot of classes, if everyone is patient enough. Note furthermore, since $\gamma(x)$, and $\delta(y)$ are strictly increasing, class borders are indeed defined by the lowest type.

[23]Burdett and Coles (1997) show that these two adjustments can be replaced by lowering the upper bound of integration in the continuous time value function, *if* it is written in the form of 1.
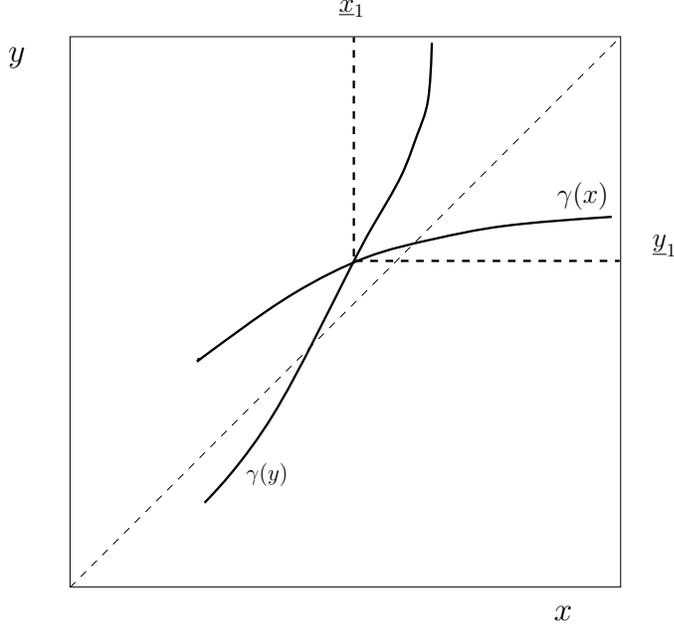
Figure 3: Intersection of the two acceptance functions

• STEP 4: Proceed inductively, by creating enough space between actions such that the highest type of the lower class would prefer to be in the higher class if the signal of the next group were any 'closer' to his current signal. Conversely, the lowest member of the higher class weakly prefers to be in his class. By lemma 5, and in particular corollary 2, if $V(x_i^c, a_i) - V(x_i^c, a_{i+1}) = c(a_i - a_{i+1})$, then $V(x, a_i) - V(x, a_{i+1}) < c(a_i - a_{i+1})$, for all $x < x_i^c$. This defines a sequence $\{a_{mI}, \ldots, a_{m1}\}$, and $\{a_{wI}, \ldots a_{w1}\}$, supposing everyone is finding a matching partner.

• STEP 5: Formulate the off-equilibrium path beliefs and responses. In this equilibrium, we can hold any belief about deviators to an action lower than a_{mi}, as long as they are members of the classes $i+1, \ldots, I$. In particular, we can believe that the deviators are the highest types from these classes. Since we are working with the continuum, we loosely write this as believing that they are the lowest type of class i, but *they will not be accepted*, which is weakly optimal. More accurately, the highest members of the previous class $i-1$ would be types arbitrary close to, but not equal to, the lowest type of class i; and the agents on the other side would *strictly* prefer not to accept them. Thus, with a little hand waving, we write $G(x_i^c | a) = 1$, for $a_{m,i+1} < a < a_{m,i}$, and the optimal acceptance decision actually is a reservation policy with respect to *all* actions. ∎

Claim 1 *This equilibrium survives D1. No equilibrium with a class smaller than given by (20) can be followed by another class and survive D1.*

Proof. Let us look at the equilibrium described above. For any action greater than the highest action played, a_1, any belief is acceptable, since no person will ever do even weakly better than playing a_1. We will focus on the actions between the action a_2 and a_1. The argument for all lower actions follows identical lines. We believe that, for $a_2 < a < a_1$,

$$G(x|a) = \begin{cases} 0 & \text{if } x < x_1 \\ 1 & \text{if } x \geq x_1 \end{cases} \tag{21}$$

Note that, we assume that this belief implies that none of the first class will accept the deviator. The probability distribution in 21 is, as explained in step 5, convenient shorthand (with a little waving) for believing that the deviator is the highest type of the class below, whom members of the class in question find it strictly optimal to reject. Now, we still have to show that this belief survives the D1. This boils down to showing that there does not exist a type x' such that, for a reply r, denoting an acceptance interval $[\underline{y}, y^r]$ of the (ex ante) receiver[24], *if $V(a, r, x) \geq \hat{V}(x)$, then $V(a, r, x') > \hat{V}(x')$.* There are two possibilities: $x' < x_1$ and $x' > x_1$.

• Case 1: we concentrate on $x_2 \leq x < x_1$ (the lower x's follow by a completely analogous reasoning). Note, since r is identical, both have the same acceptance sets. And, we care only about the net gain of the deviation (increased acceptance of types that we will accept) minus the net cost. Let y^r be the highest type that accepts them, as a response. Consider the case that y^r is such that net expected gain

$$\alpha\beta \int_{y_1}^{r} u(x_1, y) dF_w(y) - c(a - a_2) = 0 \tag{22}$$

then, for $x' < x_1$, it must hold that

$$\alpha\beta \int_{y_1}^{r} u(x', y) dF_w(y) - c(a - a_2) < 0$$

since $\frac{u(x_1, y)}{u(x_1, \underline{y})} > \frac{u(x', y)}{u(x', \underline{y})}$, and $u(x_1, \underline{y}) > u(x_1, \underline{y})$ implies $u(x_1, y) > u(x', y)$ for all $y > \underline{y}$.

• Case 2: $x_1 < x' < \bar{x}$. Same reasoning. Take y^r to satisfy (22). But we know, by construction x_1[25] is indifferent between playing a_2 and a_1. It must then also hold that

$$c(a_1 - a) - \alpha\beta \int_{\bar{y}}^{y^r} u(x_1, y) dF_w(y) = 0$$

However, a higher type's net gain or loss, is

$$c(a_1 - a) - \alpha\beta \int_{\bar{y}}^{y^r} u(x', y) dF_w(y) < 0$$

---

[24] D1 is defined for any response r, not interval responses. However, by the outside option of the sender, the only relevant receiver actions are those involving $y \geq y_2$ accepting. In the case of $y < y_2$, the sender will simply take his outside option. To have intervals above y_2 is done for ease of exposition, but any other positive mass combination of types will do.

[25] Again, we are hand waving a bit here, since really we are concerned with a type infinitesimally smaller than x_1; however, the line of the argument is the same.

by the same argument as above, since $x_1 < x'$, $\int_{\bar{y}}^{y^r} u(x', y)dF_w(y) > \int_{\bar{y}}^{y^r} u(x_1, y)dF_w(y) = 0$. Repeating these arguments establishes that there does not exists any type x', for which $V(a, r, x_1) \geq \hat{V}(x_1)$, implies $V(a, r, x') > \hat{V}(x')$. Hence, at $a_2 < a < a_1$, $x_1 \in \mathbb{X}$, so the belief spelled out above does not violate D1. Indeed, intuitively x_1 is the one who has most to gain from deviating. We can similarly proceed to establish that the similarly chosen beliefs between the other classes also satisfy D1. Hence, for any $a \neq A^{eq}$, D1 is not violated, and hence this equilibrium survives D1.

Now suppose the top class is smaller than specified in proposition 4. We have established that between two classes, we believe it is the highest type of the lower class who is the deviator. However, this means that for a deviation $a_2 + \varepsilon$, the deviator gets a discrete rise in payoff, since now a positive interval of types of class 1 individuals will accept him. Hence, this equilibrium breaks down. ∎

4.4 Full Relevant Separation Equilibrium

(INCOMPLETE) We are also interested whether frictions allow us to get the equivalent of the separating equilibrium. Here, we call it full relevant separation, as there is one group that will always be treated as equivalent in the equilibrium with perfect observability: those that are always accepted. For them there is no need to distinguish themselves, if everybody else is playing different actions. With full relevant separation, we get the same matches as in the perfect observable case.
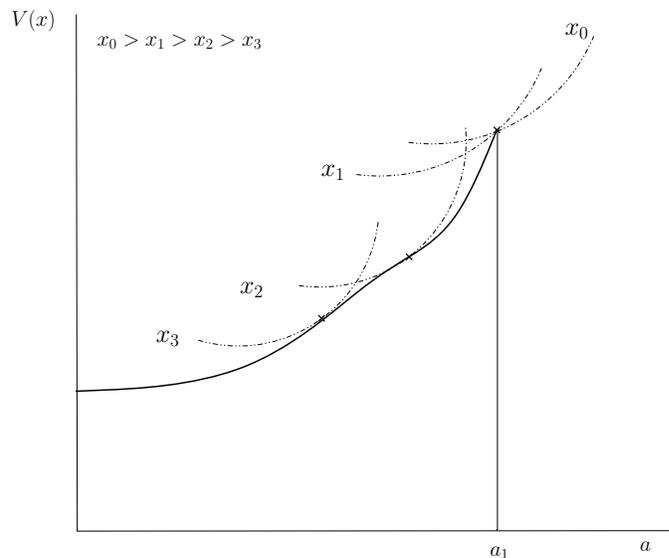

Figure 4: Separation in Equilibrium

Seeing that this is in fact an equilibrium is not completely trivial, as with standard assumptions, separating equilibria in signaling games with a continuum of types need

and imply differentiable action functions (Mailath 1987), in an environment with twice differentiable payoff functions. Here our environment is slightly different, as the payoff functions of posing as a different type follows the shape of the value function: with kinks in it, as stated in proposition 2.

Let us define the following value function

$$\widehat{V}(x, \hat{x}) = \int_{\mu(\hat{x}, x)} \big(u(x, y) - \widehat{V}(x, \hat{x}) \big) dF_w(y) \tag{23}$$

where $\mu(\hat{x}, x)$ is the matching set of x, when he gets acceptance based on being considered $\hat{x}$, and the following payoff function

$$U(x, \hat{x}, a) = \widehat{V}(x, \hat{x}) - ca \tag{24}$$

We can derive the following properties of $\widehat{V}(x, \hat{x})$:

Conjecture 1 $\widehat{V}(x, \hat{x})$ *is supermodular and a.e. differentiable (same reasoning as lemma 4)*

With this, we get the following currently unproven (and therefore not necessarily correct) conjecture:

Conjecture 2 *An equilibrium with full relevant separation exists. Let $\hat{x} = \Delta(\bar{y})$. Then the actions chosen by each type up to $\hat{x}$ are given implicitly by $\frac{\widehat{V}_2(x, x)}{c} = a'(x)$ in this equilibrium*

First idea of Proof. A.e. differentiable $\widehat{V}(x, \hat{x})$ implies that left and right derivatives exist. Now we can map, WLOG, types x on interval $[x_1, x_2]$ into a new interval $[x_1^*, x_2^*]$, such that matches are still identical, but differentiability is restored. Absence of for certain properties normally assumed on the continuous second derivatives can be compensated(?) by supermodularity of the value function and the properties that carry over from the twice differentiability of utility function(?). ■

The idea of the proof can be shown in figure 4. Even though there are kinks in the value function, we can also have corresponding kinks in the equilibrium $\alpha_m(x)$ function, such that in the equilibrium, the graph of $\widehat{V}(x, x)$ and $\alpha_m(x)$ is smooth.

4.5 Comments on the Equilibria

Both equilibria above satisfy the D1 criterion. Note that, in this context with the strict single-crossing property, D1 implies that strategies need to be in the form of a reservation property with respect to all actions. This means that higher actions off the equilibrium path will never be punished, but only treated at least as well as the 'previous' equilibrium action. Hence, we have that pooling in this context does not in any way depend on 'harsh' punishments off the equilibrium path.

In the perfect segregation equilibrium, a man, say, can deviate to a higher off-equilibrium path action, but since the reservation *action* of the entire class (of women) above the class he is currently accepted by is the next, more expensive, equilibrium action played, there is no gain of doing so. We can see this equilibrium in figure 5. Type x_1^c is indifferent between the two equilibrium actions a_2, a_1, and there is no gain through increased acceptance (the horizontal dashed line) for him in playing an action between a_2 and a_1.
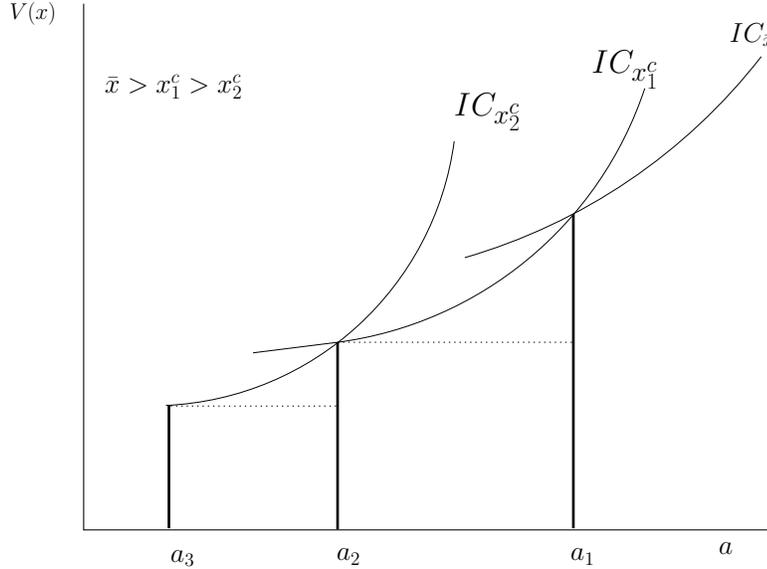

Figure 5: Scaled Indifference Curves

In the separation equilibrium (see figure 4), on the other hand, for a type that is not accepted by all, increasing the signaling action by a bit will lead to strictly better acceptance[26]. In this equilibrium, everybody but the very highest types who are accepted by all weigh off increased acceptance and increased costs. Again, everybody has an acceptance policy in reservation actions, but now this action differs for every type (except those that accept everybody).

The multiplicity of equilibria (after the D1 refinement) bears some relation to the multiplicity in Burdett and Coles (2001) in the case of investment in self-improvement. In their 'Californian' equilibrium there is pressure to self-improve, because everybody does it; in their 'Scottish' equilibria, there is not much self-improvement undertaken by others, and hence no strong need to improve oneself. In our case, in the separating equilibrium, an agent is forced to separate herself from her immediate neighbors in quality, since not doing so leads to worse acceptance: the men can now infer that she is not of the high(er) quality neighbors that have separated themselves. Thus, in

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[26]With a caveat for the highest separating type, for which this is not necessarily true.

some sense, when others separate, one is forced to improve one's signal as well (and separate as a result). In the segregation equilibrium, on the other hand, it is sufficient to be just as good as one's neighbors. There are differences between our setup and results and Burdett and Coles', though. They cover the case of an investment in additional quality; what matters is the total quality after investment. Everyone starts with different initial qualities. In our case, everybody starts with zero amount of the signal. What is more, the signal is irrelevant for the utility of a match, in the end. In their case, self-improvement in quality is done where a group of people choose the same (threshold-values) in quality, while in our case separation can be over almost the entire range of types.

An important role in our setup is played by the fact that signals are played purely to affect acceptance decisions. If one is accepted by all at a certain signal, there is no need to deviate to higher signaling actions anymore. Thus, given that one believes that one's class members (of the highest class) are not separating themselves, one will be accepted when one plays as well as the others. The acceptance decisions are one of the factors that allows the perfect segregation equilibrium to arise, and to survive D1.

As such there are more symmetric equilibria that survive D1, but they are all combinations of the two types of equilibria[27]. What happens in a perfect segregation equilibrium is that people in a class only match with themselves, and as such, become totally irrelevant for all other people (except that for those people, the effective meeting rate has decreased). However, the remaining people could be in full relevant separation equilibrium with each other. Hence, we can have a (few) class(es) on top, and the remaining people separating. However, the opposite pattern cannot happen when we use the D1 refinement: once the highest set of people are playing separating strategies, the remaining people cannot form pooling groups below. The intuition for this is simply that the highest type of the pooling group is able to reveal himself, and will be better treated (at a gain), since more of the higher people will accept him, upon revelation.

5 An Application: Conformity in Social Interaction

The model, of course, is an extension of the marriage models with frictions, and is applicable to the cases for which these models are immediately relevant. However, we want to argue that the model potentially has wider implications and can be taken to tell us something about aspects of social interaction. In particular, the model sheds light on the issue of conformity. Below, we first want to argue that the setup of the model is appropriate to make such a claim, before giving our explanation of conformity based on the model.

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[27]Classes can be bigger than defined by equation (20). See e.g. footnote 13.

First, signaling seems to play an important role in social interaction. We are frequently judged on observable possessions of (often fairly typical) goods, like clothing, cars, types of shoes, etc, and on undertaking certain social activities, e.g. volunteering for social activities, church going, or adhering to a certain etiquette. An important reason for possessing these goods, and performing these actions, is that as a result, we expect to be treated better in our social lives. This effect is thought to happen even when owning the goods or undertaking the activity in themselves does nothing to change the things we fundamentally value in social interaction. Nonetheless, there is an obvious and rational reason for the better treatment, because those people who find it worthwhile to incur the costs for these actions are on average of 'better quality' than those who do not. In order words, the actions work as a signal for underlying fundamental values.

Moreover, frictions seem to be even more relevant for social, nonmarket situations. With the latter we mean that certain things, especially in the social sphere, cannot be simply transacted in the usual economic way. One cannot buy friendship, or, at least in most of the economically advanced societies buy one's husband or wife.

For purely economic transactions, frictions already play an often determining role, but for social interactions, this role seems to be exacerbated. For one, arbitrage —which mitigates the effect of frictions— is easier for purely economic transactions than for social transactions. Crucially for arbitrage, economic goods are usually clearly priced, and can be transferred relatively easily from person to person; with social 'goods' (such as memberships of clubs or preferential treatment) there are often no clear prices, and these goods cannot be transferred easily from person to person[28]. Another aspect of search frictions is the limited information we have about specific people, even though we do know the aggregate distribution. In practice, we know and can socially interact with only a limited number of individuals; we do meet new individuals and the new social opportunities they bring with them, but these meetings are spread out over time. The assumption of random search seems to capture this to an appropriate extent[29].

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[28] A member of a club might want to sell his club membership, however, to transfer membership the rest of the club needs to agree. This is precisely because a membership of a club is not only a 'good' for the member x, but having a person x as member is also a 'good' for the club: the two-sidedness of the productive relationship.

[29] Random search importantly means that everyone has a chance to meet everyone; it abstracts from locational considerations (i.e. having places where only the high-quality people come to meet, and places where the lower quality people meet). There are three arguments backing up the choice of random search. First, meetings never occur completely within 'classes', we always meet people that we cannot place directly in the social hierarchy. Secondly, even if we never meet a part of the population, we still can have the same need for signaling within the subpopulation of the people we actually can meet. And thirdly, random search illustrates something interesting (when used, in a sense, a bit as a counterfactual): it shows that separation in signals can occur, even when the population as such is not segregated at all. As will be argued, it shows that it could be as if people act in groups, even when the

On top of this, we believe that two-sided matching itself captures realistically social interaction. Essential in many cases of social interaction is the *mutual* agreement that is needed in order for it to have any effect. We cannot have friendship without mutual agreement; however, country club membership has the same aspect: one has to be deemed good enough for the country club, and the country club has to be good enough for oneself. (We are not alone in seeing something essential of social interaction back in simple two-sided matching, see e.g. Cole et al. 1992, 1998, 2001, and Mailath and Postlewaite 2002, 2004.)

All in all, signaling in the marriage model with frictions seems to be applicable to the case of social interaction, especially in a 'New York City' context, meaning that people are anonymous, and are judged on observable characteristics[30]. Veblen (1934, p.86-89) noted already that the anonymity of city-life raised the importance of signaling wealth; the argumentation here takes this point seriously.

Given these points, it is interesting to apply the results of the model to explain aspects of social interaction. What the model tells us is, that in an equilibrium with perfect segregation, an individual will play the same action as everyone else in his class. This clearly has a counterpart in social interaction: we often observe that people stick to 'standards of behavior', dress-codes, etiquette, and so on. In these cases 'standards' are observed to apply to a certain subset of society, but differ from subset to subset. The interaction of frictions, the two-sidedness and the need for investment might provide an additional rationale for this uniformity: even if a person investing in a different signal, will 'reveal' herself to be of a higher type among those in her class, this will neither lead to better acceptance by those who match her own class, nor to increased acceptance by higher classes. As a result he 'conforms' to the characteristic action of his class, since it is endogenously determined that higher signals give no gain (until the next class's signal), and lower signals lead to strictly decreased acceptance. Importantly, here it is the aforementioned interplay of frictions, investment choices and the two-sidedness, that yields this group behavior. Note that the resulting pattern of segregation looks like 'group-enforced' conformity, but the rationale here is entirely different. Uniformity (or conformity) within groups in our model is completely 'ex post'. In some sense, groups are only a label put on those individuals with similar actions. Interestingly, this implies that, while people do care deeply about the judgement of the people with whom they come into direct contact, *groups* as such are not *entities* which put pressure, or shape views of individuals in our model. In addition, while the pressure is from individual to individual, it is not in terms of punishment for deviation(as in Kandori 1992), but in terms of inference about (bad) quality. The argument for conformity in this paper

concept of group is entirely meaningless to those people. It is also important in the following sense: if we get segregation in a random search setting, it holds a fortiori in a more directed setting.

[30]Thus, we assume that their labels when part of their quality, are unknown.

therefore can be seen as an *economic* theory of *group identical* behavior, as opposed to a sociological theory of group identity and behavior. The latter seems more relevant for small groups *without anonymity*, but with repeated interaction and peer pressure.

This adds a new explanation to existing explanations of conformity in the literature of social economics. The explanation of conformity is not trivial in the context of rational utility maximizing individuals[31] Explanations in the literature include, for example, that people find it useful to imitate those who are thought to have better information (Conlisk 1980, Banerjee 1989), or because there are spill overs when acting together (Katz and Shapiro 1986). Bernheim (1994) considers the case that individuals care (in the utility function) about being seen as 'mainstream'; as a result they pool in the middle[32]. Another important category of explanations have been attributed precisely to arrangements within a group itself (where a group is now a exogenous, fundamental, entity), and sustained by group enforcement (Akerlof (1980) and Cole, Mailath and Postlewaite (2001) enforce conformity by the loss of social 'reputation'). These uniform actions, enforced in a social setting are often referred to as social norms, and can be sustained even in environments with meeting frictions, see Kandori 1999. The essence is that deviating behavior is punished later on; repeated interaction is therefore crucial. Since the punishment has to be credible, the social norm usually has public good or prisoner's dilemma aspects to it[33], so there is a clear benefit of acting uniformly and a clear loss when being punished.

To conclude, there are cases of uniformity where benefits of acting uniform are not as clear, since the possibilities of punishment of deviators are less obvious; this is the case we see our model addressing. Specifically, it seems that signaling seems to be an important aspect of the choice of clothes to wear, of the car to drive, of the manners showcased. In these choices, uniformity within subsets can frequently be observed. The perfect segregation equilibrium provides a rationale for this behavior. In this equilibrium a persons needs only to do as well as her peers (i.e., those persons in her subset) provide all her peers choose the same action. This reaction is optimal given the interplay of the three factors: frictions which make people accept a range of individuals, the two-sidedness which leads to considering only the acceptance decision of the other side in response to your signal, combined with investment, which you want to keep as small as possible, given the benefits of the signals.

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[31]That is, assuming we do not put a desire to conform in the utility function, which is an approach that can only lead to vacuous statements.

[32]Getting multiple pooling groups in Bernheim 1994 is difficult, and requires the existence of groups of people where each group has its own 'mainstream'.

[33]*All (at least all in the group)* agree ex ante in their preference for the uniform action, but if this uniform action is not enforced, they would prefer to cheat on the collective (i.e. group) arrangement.

6   Conclusion

In this paper we have found that perfect segregation, which in case of perfectly observable qualities was thought to depend on 'knife-edge' properties of the utility function, can be an equilibrium in more general settings, when qualities are not observable but can be signaled. We have tried to demonstrate this in a very simple setting, by making the assumptions which would not yield class-formation in the observable case, and showing that they do in the setting of this paper. Some mathematical complications in deriving the necessary technicalities for the equilibrium aside, the idea for perfect segregation itself is extremely simple: if we have a group who accepts a class of people, and does not want to match with any lower type, even if those reveal themselves to be 'best of the rest', *and* the class on the other side, in return, only wants to match with the first-mentioned class, we have a very stable situation of perfect segregation. With investment in signals this situation can arise endogenously. On the other hand, in an equilibrium where much harder competition arises among individuals, everybody is forced to distinguish themselves (except those that are accepted anyway by everyone in the observable case).

In this case, we have covered the somewhat polar case of unobservable quality. Future research could address the case where signaling can occur in a setting with imperfectly observable qualities.

Appendix A   Omitted Proofs of section 3

It is useful to define the set of agreeable matches for someone with quality x or y.

Definition 4 *The matching set $\mu(x)$ of x is the set of types willing to match with type x who are also accepted by x. I.e. $\mu(x) \equiv \{y \mid y \geq \Gamma(x)\} \cap \{y \mid x \geq \Delta(y)\}$*

Naturally, if y is in the matching set of x, x is in the matching set of y. One thing we know for sure is that frictions will lead everyone to accept a range of people. Moreover, we can find a lower bound for the size of this range (by founding an upper bound for the reservation type). This will ensure that, as long as $\underline{x} > 0, \underline{y} > 0$ the induction procedure in the following proposition will cover the entire distribution in a finite number of steps. Importantly, for reasonings below, we can recast the matching set of one type, say x, as the interval $[\Gamma(x), \Delta^{-1}(x)]$.

Lemma 6 *Matching sets for a given x or y cannot be too small, all will accept a strictly positive range of types by which they themselves are accepted; unless no positive mass of agents of the other side accepts them at all.*

Proof of lemma 6.   We can show that an upper bound for the reservation type (the lower bound of the acceptance set), is strictly and uniformly below the upper bound of the matching set for every type, and every upper bound of the matching set. Suppose the upper bound of

the matching set of a person x is $\hat{y}$. Then the value of waiting for the next period is at most $\beta u(x, \hat{y})$. where $\beta < 1$ (supposing $\alpha \leq 1$, otherwise we can rescale β). This means that, in the current period, one is happy to accept y' with $u(x, y') = \beta u(x, \hat{y})$ By the properties of u (assumption 1), we can find a y' for every $x, \hat{y}$.

This basically defines an (implicit) function $y'(x, \hat{y})$ from $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$ into $[0, \bar{y}]$. This function is continuous. Then the function $k : \bar{y} - y'(x, \hat{y})$ defined on the same domain has a minimum, which we denote by κ. The reasoning above tells us that $\kappa > 0$. This means that no matter which type we are, or which is the highest type y that accepts us, we will always accept a type $y - \kappa$. Note, if the highest type that accepts us, is $\underline{y}$, we would like to accept type $\underline{y} - \kappa$, but there is no such type around.

Note that we have found a lower bound on the range of types which are accepted for any type x, for any $\hat{y}$, and for any distribution of types. ■

Proposition 2 (repeated). *Assume assumptions 1, and 2. An equilibrium exists (Eeckhout 1999). In an equilibrium the acceptance function $\Gamma(x), \Delta(y)$ is continuous, strictly increasing.*

Proof of proposition 2. We now construct an inductive procedure (as done previously in Eeckhout 1999) that allows us to find $\Gamma(x)$ and $\Delta(y)$. We then prove that these are continuous and strictly increasing. Define the following mapping of $V \in \mathbb{R}$ into $\mathbb{R}$ (where type x and upper bound $\bar{y}(x)$ as taken to be exogenously given *parameters*)[34]

$$TV_{x,y(x)} \equiv \alpha\beta \int_{\underline{y}}^{\bar{y}(x)} \max\left\{u(x,y), V_{x,y(x)}\right\} dF_w(y) + (1-\alpha)\beta V_{x,y(x)} \tag{25}$$

The established properties of lemma 3 still apply, $TV_{x,y(x)}$ is a contraction, and $V^n_{x,y(x)}$ converges monotonically towards the fixed point. Thus given the set of women accepting the man in question, we can find his reservation type. (Perfectly analogous we can find the reservation type of each woman with type y, given the highest type of man that still accepts her.). Moreover, if we compare the results of having two different upper bounds, say $y_1(x) < y_2(x)$, then we can make the following reasoning: Take $V^*_{x,y_2(x)}$, the fixed point of (25), with upper bound $y_2(x)$. Now, take $V_{x,y_1(x)} = V^*_{x,y_2(x)}$. Since

$$TV_{x,y_1(x)} = V^*_{x,y_2(x)} - \int_{y_1(x)}^{y_2(x)} u(x,y) dF_w(y) < V^*_{x,y_2(x)} = V_{x,y_1(x)}$$

by monotonicity, we have $V^*_{x,y_1(x)} < V^*_{x,y_2(x)}$. Thus a strictly lower upper bound of the set of accepting types, leads to strictly lower value function, and therefore (by monotonicity of u), a lower reservation type. Now, we use this to derive the properties of $\Gamma(x), \Delta(y)$

STEP 1: We derive the reservation types conditional on the fact that they are accepted by everyone on the other side. I.e., we derive $\gamma(x), \delta(y)$. Let x_1 denote $\delta(\bar{y})$ (similarly for $y_1 = \gamma(\bar{x})$). Then for $x_1 \leq x \leq \bar{x}$, $\Gamma(x) = \gamma(x)$; likewise, we derive $\Delta(y)$ for $y_1 \leq y \leq \bar{y}$. The properties from γ, δ carry over, so $\Gamma(x)$ and $\Delta(y)$ are continuous and strictly increasing.

STEP 2A: Now, there exists a second interval of $x_2 \equiv \Delta(y_1) \leq x < x_1$, for which the upper bound of the matching set is well-defined, and given by $y(x) = \Delta^{-1}(x)$. Thus for these types

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[34]To call the V, $V_{x,y(x)}$, and the fixed point of this mapping, given parameters, $V^*_{x,y(x)}$, constitutes an abuse of notation, but we hope the meaning is clear.

a reservation type exists. Moreover, for $x_1 > x' > x \geq \Delta^{-1}(y_1)$, it must hold that for identical upper bounds $\bar{\bar{y}}$, $V^*_{x',\bar{\bar{y}}} > V^*_{x,\bar{\bar{y}}}$, and since $\Delta^{-1}(x') > \Delta^{-1}(x)$, we must have that

$$V^*_{x',\Delta^{-1}(x')} > V^*_{x,\Delta^{-1}(x)}$$

By the monotonicity of utility, and lemma 2, it must be that $\Gamma(x') > \Gamma(x)$. Hence over this second interval $\Gamma(x)$ is also strictly increasing.

We also have to establish that $\Gamma(x)$ is continuous, both between x_2 and x_1, as well as at x_1. Note that over the entire interval $[x_2, \bar{x}]$, $\Delta^{-1}(x)$ is a continuous function. Suppose $\Gamma(x)$ is discontinuous at $\hat{x}$. Then it can only jump upwards. Given that $\Gamma(x)$ is derived from an optimal response, it must be at least that the value function also jumps up at $\hat{x}$. Then there exists a $\eta > 0$, such that $V^*_{\hat{x},\Delta^{-1}(\hat{x})} - V^*_{x,\Delta^{-1}(x)} > \eta > 0$, $\forall\, x < \hat{x}$ However, the following function is continuous ($\Omega \equiv \frac{\alpha\beta}{(1-\beta+\alpha\beta)}$:

$$\mathcal{V}(x) \equiv \Omega\left(\int_{\Gamma(\hat{x})}^{\Delta^{-1}(x)} u(x,y)dF_w(y) + u\big(x,\Gamma(\hat{x})\big)F_w\big(\Gamma(\hat{x})\big) \right) + \tag{26}$$

This is the value for type x, of copying $\hat{x}$'s acceptance policy. As this function is continuous in x, and $\mathcal{V}(\hat{x}) = V^*_{\hat{x},\Delta_{-1}(\hat{x})}$, there exists an $\varepsilon > 0$, such that for $\hat{x} - \varepsilon$, $V^*_{x-\varepsilon,\Delta^{-1}(x-\varepsilon)} > \mathcal{V}(x-\varepsilon) > \mathcal{V}(\hat{x}) - \eta = V^*_{\hat{x},\Delta^{-1}(\hat{x})} - \eta$. This implies that $\Gamma(x)$ is continuous and strictly increasing over $[x_2, \bar{x}]$

STEP 2B: Now, we can do the same thing for the women, and find $\Delta(y)$ between $[y_2, y_1]$.

STEP 3: We proceed inductively, and use $\Delta(y)$, $y_2 \leq y < y_1$, to find $\Gamma(x)$, $x_3 \leq x < x_2$. We can use $\Gamma(x)$ defined on $x_2 \leq x < x_1$, and $x_3 \leq x < x_2$, to find y_3, y_4, and $\Delta(y)$, for respectively $y_4 \leq y < y_3$ and $y_3 \leq y < y_2$ (The functions are continuous at all these points, we can use the same argumentation as in step 2a). Induction stops (after solving out the remain reservation type function on the other side), when we hit on an x with $\Gamma(x) = \underline{y}$, or a y, with $\Delta(y) = \underline{x}$. All x below the x where $\Gamma(x) = \underline{y}$, accept everybody (so, wlog, can be seen as all having $\Gamma(x) = \underline{y}$). (similar, with $\Delta(y)$. Technically, $\underline{x}$ does not accept types below $\Gamma(\underline{x})$, but it does not matter to act as if did, because now on is only accepted by a mass zero of agents, which one has zero probability of meeting.

There are two issues: we have to make sure that we can always *continue* the procedure until we have all the information we need. And, we have to make sure that the process does give us all the information we need after a certain number of iterations.

• This is shown by the following two issues:

1) Note that by the strict increasingness of $\Gamma(x), \Delta(y)$ and lemma 6, there can never be a set of points (x, y), such that $\Gamma(x) = y$ while simultaneously $\Delta(y) = x$ (i.e. $\Delta^{-1}(\Gamma(x)) = x$). Thus, the process never stops prematurely, in finite time.

2) Next, we have to rule that we do not get to the 'end' in infinite time. To show: if we are not at the end, we will always be able to gain information on a interval on types with length uniformly bounded from zero. Suppose that we have a point x', and have found $\Gamma(x)$ for $\bar{x} \geq x \geq x'$, and $\Delta(y)$, for $\bar{y} \geq y > \Delta^{-1}(x')$. Given that $\Gamma(x)$ and $\Delta(y)$ are strictly increasing and continuous on these intervals, we now can derive a continuous and strictly increasing $\Delta(y)$ up to $y' = \Gamma(x')$. However, this gives us the opportunity to derive $\Gamma(x)$, up to $x'', x'' < x'$, where $x'' = \Delta(\Gamma(x'))$. Now, importantly, by lemma 6, $|\Delta(\Gamma(x')) - x'| > \kappa > 0$ (i.e. uniformly bounded away from zero), in any common norm, for any x'. This means that if we know $\Delta(y)$, and $\Gamma(x)$ above x' and

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$\Delta^{-1}(x')$, we can derive $\Gamma(x)$ on $[x'', x']$, with $|x'' - x'| > \kappa$, and $\Delta(y)$, between $[\Delta^{-1}(x'), \Gamma(x')]$. We can repeat the inductive step (step 2a, 2b) and always gain new information on a interval of types of at least length $\kappa > 0$ (on both sides!), till we have all information we need. We are done with deriving $\Gamma(x)$, when we have found $\Gamma(x) = \underline{y}$; similarly when we found $\Delta(y) = \underline{x}$. In case we hit on an x with $\Gamma(x) = \underline{y}$, every lower x has $\underline{y}$ as reservation type (similarly, if we hit first on a y with $\Delta(y) = \underline{x}$). We now also have all information we need to completely solve the other side's reservation decisions in one step. Thus, this inductive method always ends in finite steps. Moreover, by repeating the argument in step 2a about strictly increasing and continuous reservation type functions, we can immediately show that the Γ, Δ -functions are strictly increasing and continuous over the entire domain. Note, in case the two sides of the model are not identical, it can be the case that some types on *one and only one* side of the market remain unmatched. ∎